MATH 245 S22, Exam 2 Solutions

1. Carefully state the following theorems: Proof by Contradiction Theorem, Nonconstructive Existence Theorem

   The Proof by Contradiction Theorem says that, for any propositions p, q, if $p \wedge \neg q \equiv F$, then $p \to q$ is true. The Nonconstructive Existence Theorem says that, if $(\forall x \in D,\ \neg P(x)) \equiv F$, then $\exists x \in D,\ P(x)$ is true.

2. Carefully define the following terms: Proof by Shifted Induction, Big Omega (Ω)

   For some $s \in \mathbb{Z}$ and some predicate $P(x)$ (with domain $\mathbb{Z}$), to prove $\forall x \in \mathbb{Z}$ with $x \geq s$, $P(x)$ by Shifted Induction, we must (a) prove $P(s)$; and (b) prove $\forall x \in \mathbb{Z}$ with $x \geq s$, $P(x) \to P(x+1)$. Given two sequences a_n and b_n, we say that $a_n = \Omega(b_n)$ if $\exists n_0 \in \mathbb{N}$, $\exists M \in \mathbb{R}$, $\forall n \in \mathbb{N}$ with $n \geq n_0$, we have $M|a_n| \geq |b_n|$.

3. Let $x \in \mathbb{R}$. Use cases to prove that $|x-1| + |x+1| \geq x$.

   METHOD 1: We break into three cases, based on whether $x < -1$, $-1 \leq x \leq 1$, or $1 < x$.
   Case $x < -1$: $|x-1| + |x+1| = -(x-1) - (x+1) = -2x \geq 2 > -1 > x$.
   Case $-1 \leq x \leq 1$: $|x-1| + |x+1| = -(x-1) + (x+1) = 2 \geq 1 \geq x$.
   Case $1 < x$: $|x-1| + |x+1| = (x-1) + (x+1) = 2x > x$. (the last since $x + x > x + 0$).

   METHOD 2: We break into three cases, based on whether $x < 0$, $0 \leq x \leq 1$, or $1 < x$.
   Case $x < 0$: $|x-1| + |x+1| \geq 0 > x$. (the first since every absolute value is ≥ 0).
   Case $0 \leq x \leq 1$: $|x-1| + |x+1| \geq 0 + (x+1) \geq x$. (the first since $|x-1| \geq 0$).
   Case $1 < x$: same as in method 1.

4. Prove or disprove: $\forall x \in \mathbb{Z},\ !y \in \mathbb{Z}\ \ x = y^3$.

   The statement is true. Let $x \in \mathbb{Z}$ be arbitrary. Suppose $y, z \in \mathbb{Z}$ with $x = y^3$ and $x = z^3$. Now $y^3 = x = z^3$. Taking cube roots, we get $y = z$. [Note that cube roots are unique in $\mathbb{R}$]

5. Prove that for all $n \in \mathbb{N}$, we must have $\sum_{i=0}^{n}(2i-1) = n^2 - 1$.

   Proof by vanilla induction.
   Base case is $n = 1$: $\sum_{i=0}^{1}(2i-1) = (2 \times 0 - 1) + (2 \times 1 - 1) = 0$, while $1^2 - 1 = 0$.
   Now, let $n \in \mathbb{N}$ be arbitrary, and assume that $\sum_{i=0}^{n}(2i-1) = n^2 - 1$. Adding $2n+1$ to both sides, we get $2n + 1 + \sum_{i=0}^{n}(2i-1) = n^2 - 1 + 2n + 1$. We simplify to get $\sum_{i=0}^{n+1}(2i-1) = (n+1)^2 - 1$.

6. Use (some form of) induction to prove that for $n \geq 1$, all Fibonacci numbers F_n are positive.

   The proof must use strong induction, and needs two base cases.
   Base case $n = 1$, $F_1 = 1 > 0$. Base case $n = 2$, $F_2 = F_1 + F_0 = 1 + 0 = 1 > 0$.
   Now, let $n \in \mathbb{N}$ with $n \geq 3$ be arbitrary, and assume that F_{n-1} and F_{n-2} are both positive. We have $F_n = F_{n-1} + F_{n-2}$, and the sum of two positive numbers is positive.

7. Solve the recurrence with initial conditions $a_0 = 3$, $a_1 = -1$ and relation $a_n = a_{n-1} + 6a_{n-2}$ (for $n \geq 2$).

   This relation has characteristic polynomial $r^2 - r - 6 = (r-3)(r+2)$. We have two distinct roots, so the general solution is $a_n = A3^n + B(-2)^n$. Our initial conditions give $3 = a_0 = A3^0 + B(-2)^0 = A + B$, and $-1 = a_1 = A3^1 + B(-2)^1 = 3A - 2B$. The system of equations $\{3 = A + B, -1 = 3A - 2B\}$ has unique solution $A = 1, B = 2$, so our recurrence has specific solution $a_n = 3^n + 2(-2)^n$.

8. Let $a_n = n^{1.9} + n^{2.1}$. Prove or disprove that $a_n = O(n^2)$.

The statement is false. Let $n_0 \in \mathbb{N}$ and $M \in \mathbb{R}$ be arbitrary. Set $n = \max(n_0, \lceil M^{10} \rceil + 1)$. This choice of n guarantees that $n \geq n_0$ and that $n > M^{10}$. Taking the tenth root, we get $n^{0.1} > M$. Multiplying by n^2, we get $n^{2.1} > Mn^2$. Now, we have $|a_n| = |n^{1.9} + n^{2.1}| = n^{1.9} + n^{2.1} > n^{2.1} > Mn^2 = M|b_n|$.

9. Prove that for all $x \in \mathbb{R}$ with $x \geq 0$, we must have $\lfloor x \rfloor^2 \leq \lfloor x^2 \rfloor$.

Let $x \in \mathbb{R}$ with $x \geq 0$. By definition of floor, $\lfloor x \rfloor \leq x$. By Theorem 5.18 in the book, $\lfloor x \rfloor \geq \lfloor 0 \rfloor = 0$. We multiply both sides by $x \geq 0$ to get $\lfloor x \rfloor x \leq x^2$, and by $\lfloor x \rfloor \geq 0$ to get $\lfloor x \rfloor^2 \leq \lfloor x \rfloor x$. Combining, we get $\lfloor x \rfloor^2 \leq x^2$. We now apply Theorem 5.18 from the book to get $\lfloor \lfloor x \rfloor^2 \rfloor \leq \lfloor x^2 \rfloor$. Lastly, since $\lfloor x \rfloor^2 \in \mathbb{Z}$, we apply Theorem 5.19 from the book to get $\lfloor \lfloor x \rfloor^2 \rfloor = \lfloor x \rfloor^2 + \lfloor 0 \rfloor = \lfloor x \rfloor^2$. Putting it all together, we get $\lfloor x \rfloor^2 \leq \lfloor x^2 \rfloor$.

10. Consider the recurrence with initial conditions $T_0 = 0$, $T_1 = 0$, $T_2 = 1$ and relation $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ (for $n \geq 3$). Prove that, for all $n \in \mathbb{N}_0$, we have $T_n < 2^n$.
DO NOT TRY TO SOLVE THE RECURRENCE.

This question is similar to Thm 6.13. The proof must use strong induction, and needs three base cases: Base case $n = 0$, $T_0 = 0 < 1 = 2^0$. Base case $n = 1$, $T_1 = 0 < 2 = 2^1$. Base case $n = 2$, $T_2 = 1 < 4 = 2^2$.

Now, let $n \in \mathbb{N}$ with $n \geq 3$, and assume that $T_{n-1} < 2^{n-1}$, $T_{n-2} < 2^{n-2}$, and $T_{n-3} < 2^{n-3}$. We have $T_n = T_{n-1} + T_{n-2} + T_{n-3} < 2^{n-1} + 2^{n-2} + 2^{n-3} < 2^{n-1} + 2^{n-2} + 2^{n-3} + 2^{n-3} = 2^{n-1} + 2^{n-2} + 2^{n-2} = 2^{n-1} + 2^{n-1} = 2^n$.

NOTE: These are called "Tribonacci numbers". To solve the recurrence, one would need to find the nasty-ass roots of the characteristic polynomial $r^3 - r^2 - r - 1$ (which can be done, with some advanced methods).